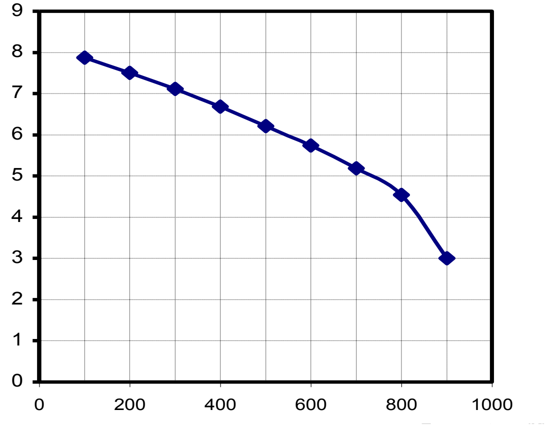
2. HEAT TRANSFER AND THERMAL STRESSES

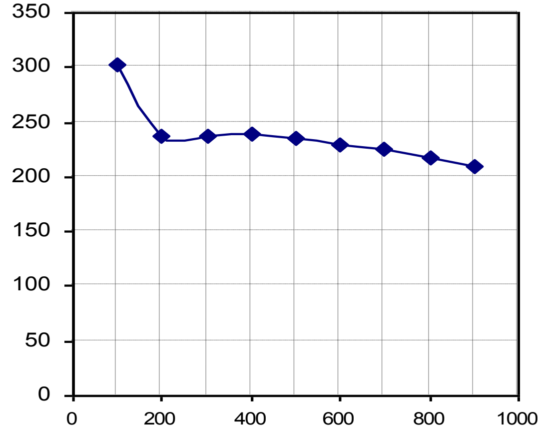
Temperature may influence the strength of a structure by:

- thermal expansion effect (thermal stresses)
- impact of the temperature on the mechanical properties of materials

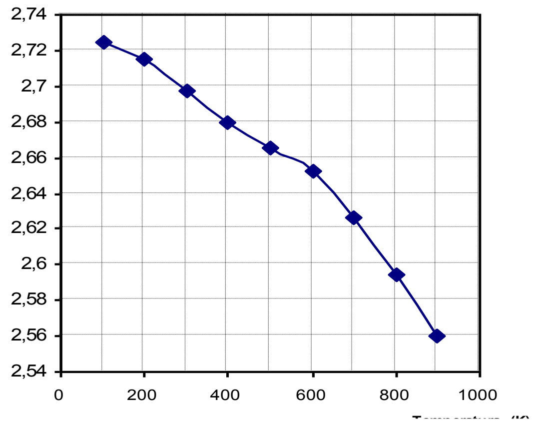
Young's modulus E (x10^4 MPa)



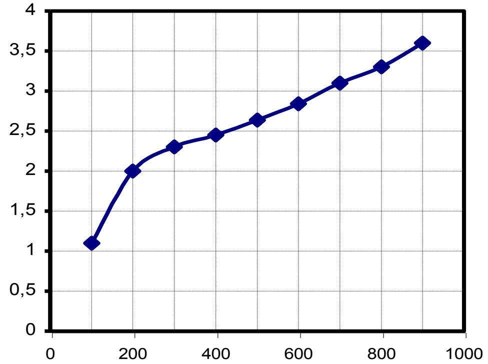
thermal conductivity λ (W/(mK))



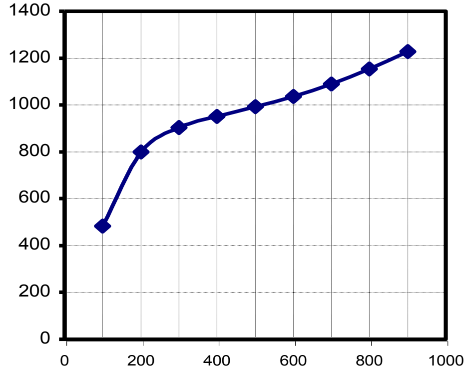
density ρ (kg/m^3)



thermal expansion coefficient α (1/K)



specific heat c (J/(kgK))



Thermo-mechanical properties of aluminium (properties vs temperature in K)

Thermal expansion coefficient

$$\alpha(T)=\frac{l(T)-l(T_{ref})}{(T-T_{ref})\,l(T_{ref})}=\frac{\Delta l}{l\,\Delta T}=\frac{\varepsilon(T)}{\Delta T}$$

$$\{\varepsilon\}=\{\varepsilon\}_T+\{\varepsilon\}_s \qquad \begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}=\begin{Bmatrix}\alpha_x\Delta T\\ \alpha_y\Delta T\\ \alpha_z\Delta T\\ 0\\ 0\\ 0\end{Bmatrix}_T+\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}_s$$

Thermal stresses may be caused by

- non uniform temperature field
- temperature change and nonhomogeneous materials
- temperature change and statically indeterminate constraints (reactions)

Total strain vector is the sum of thermal strain and elestic strain vectors.

(Hooke's law $\{\sigma\}=[D]\{\varepsilon\}_s$!)

T_0 – reference temperature for an unstrained state, in isotropic case $\alpha_x=\alpha_y=\alpha_z=\alpha$

Example – simple relation temperature-stress in the case of statically indeterminate constraints

The simple bar fixed at both ends

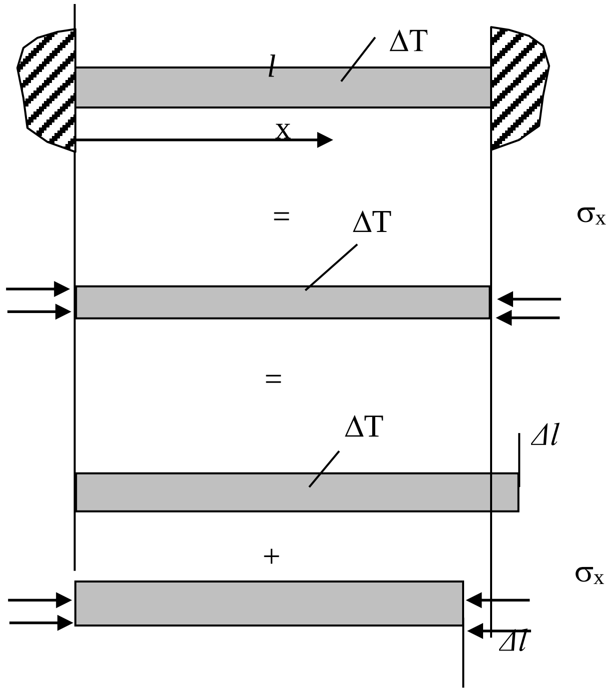
Loads

=

change of the temperature ΔT

+

compression



The elongation equals 0 → longitudinal strain $\varepsilon_x = 0$

$$\varepsilon_x=\varepsilon_{xT}+\varepsilon_{xs}=0\,,$$

$$\varepsilon_{xT}=\alpha\Delta T\,,\ \varepsilon_{xs}=\sigma_x/E\,.$$

$$\varepsilon_{xs}=-\varepsilon_{xT}=-\alpha\Delta T$$

$$\sigma_x=E\varepsilon_{xs}=-E\alpha\Delta T$$

For typical steel

($E=2\times10^5$ MPa, $\nu=0{,}3$, $\alpha=1{,}2\times10^{-5}$ 1/°C) and

$\Delta T=100$°C the result is $\sigma_x=240$ MPa

The standard approach in thermal stresses analysis using FEM:

A. The heat flow analysis (steady state or transient)

B. The stress analysis using the current temperature field as a kind of body forces

A.THERMAL ANALYSIS

Partial differential equation describing transient heat flow through a solid (law of conservation of energy):

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial T}{\partial z}\right) + q_v(x,y,z,t),$$

isotropic case $$\frac{\partial T}{\partial t} = a_d\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + q_v = a_d \nabla^2 T + q_v,$$

where $\alpha_d = \lambda / c\rho$ is the thermal diffusivity

$T(x,y,z,t)$ – temperature,

q_v – int. heat generation rate per unit volume(W/m^3),

$\lambda_x, \lambda_y, \lambda_z$ – heat conductivity coefficients(W/mK),

ρ – density (kg/m^3),

c – specific heat (J/kg).

Thermal properties of selected materials at 20°C (RT)

Material	Thermal expansion coefficient α (1/^{0}C)	Thermal conductivity λ (W/mK)	**Specific heat c (J/kgK)**	**Density ρ (kg/m^3)**
Copper	$1,7\cdot10^{-5}$	390	400	9000
Aluminium	$2,4\cdot10^{-5}$	210	900	2700
Pine wood	$0,4–0,6\cdot10^{-5}$	0,1–0,5	1300–2700	500–700
Steel 1H13	$1,1\cdot10^{-5}$	29	440	7700
Glass	$0,05–0,09\cdot10^{-5}$	0,7–1,3	600–800	2500
Rubber	$7,7\cdot10^{-5}$	0,16	1400	1200

Three modes of heat transfer: conduction, convection, radiation

Conduction

Is the transfer of thermal energy through the solid body or fluid due to the temperature gradient.

The equation describing this heat transfer is Fourier's law.

For an isotropic medium: $\bar{q} = -\lambda \,\mathrm{grad}(T)$

where $\bar{q}$ is the rate of heat flow per unit area (heat flux) and λ is the thermal conductivity.

Convective Heat Flow

The transfer of thermal energy between the solid object and its environment due to fluid motion.

The rate of heat flow across a boundary is proportional to the difference between the surface temperature and the temperature of adjacent fluid

$$q = \alpha_k (T_0 - T_c) \quad \text{(Newton's law)}$$

where α_k is the convection coefficient (film coefficient)

Typical magnitudes of convection coeeficient (W/(m^{2}K))

Medium (fluid)	Free convection	Forced convection
gas (air)	5–30	30–500
water	30–300	300–20000
oil	5–100	30–3000
liquid metals	50–500	500–20000

Radiation Heat Exchange

Stefan – Boltzman's law:

$$e = \varepsilon \sigma_0 T^4 = C T^4,$$

$$\sigma_0 = 5{,}67 \cdot 10^{-8} \ \mathrm{W/m^2K^4}$$

ε emissivity of the surface ($0 < \varepsilon < 1$)

The heat exchange between two parallel surfaces

$$q_{AB} = \varepsilon_{AB} C_o \left[(T_A/100)^4 - (T_B/100)^4 \right],$$

$$C_o = 10^8 \sigma_0 ,$$

$$\varepsilon_{AB} = \frac{1}{1/\varepsilon_A + 1/\varepsilon_B - 1}$$

In computational practice heat exchange across boundary (by radiation and convection) is usually described by the convection model $q = \alpha_k (T_0 - T_c)$ where α_k is the adequate function of temperature.

The simplest case of heat flow – steady state heat transfer with constant isotropic material properties.

In that case the heat flow equation reduces to Poisson's equation: $\nabla^2 T + \mathbf{f} = \mathbf{0}$

FE method for Poisson's equation in 2D space

$$\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} + f(x_1, x_2) = 0 ,$$

where $\bar{x} = (x_1, x_2) \in \Omega$,

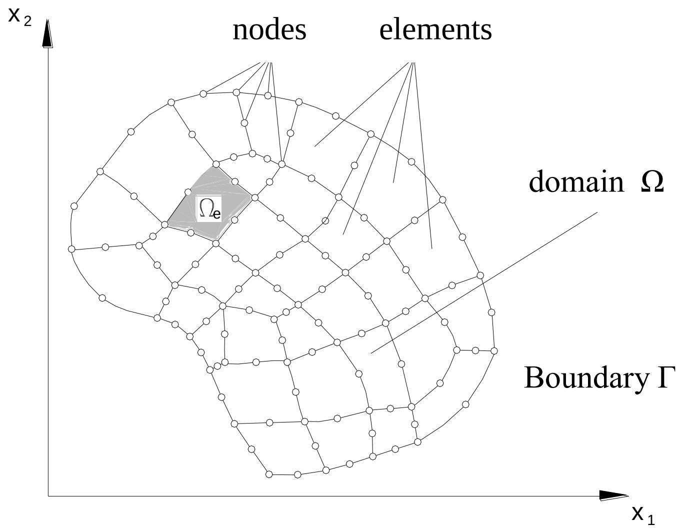
Boundary conditions

$$T(\bar{x}) = T_0 \quad , \quad \bar{x} \in \Gamma_u$$

$$q(x) = \frac{\partial T(\bar{x})}{\partial n} = q_0 \ , \quad \bar{x} \in \Gamma_q$$

(prescribed temperature or prescribed thermal flux)

Minimized functional in FEM formulation



$$I(T) = \frac{1}{2}\int_{\Omega}\left[\left(\frac{\partial T}{\partial x_1}\right)^2 + \left(\frac{\partial T}{\partial x_2}\right)^2 - 2 f(x_1, x_2) T\right] d\Omega - \int_{\Gamma_q} q_0 T d\Gamma,$$

$$\Omega = \bigcup_{i=1}^{LE} \Omega_e \qquad \text{i = 1,…, LE}$$

LE – number of the elements in the domain Ω

Approximation of the unknown function (temperature) within an element

$$T(x_1, x_2) = \sum_{i=1}^{LWE} N_i(x_1, x_2) T_i$$

T_i , i = 1,...,LWE nodal temperatures,

$N_i(x_1,x_2)$ – shape functions.

LWE – number of nodes of the element

$$I(T) \cong \sum_{i=1}^{LE} \frac{1}{2} \int_{\Omega_i} \left[\left(\frac{\partial T}{\partial x_1} \right)^2 + \left(\frac{\partial T}{\partial x_2} \right)^2 - 2 f(x_1, x_2) T \right] d\Omega_i - \sum_{j=1}^{LK} \int_{\Gamma_j} q_0 T d\Gamma_j$$

LK number of element sides on Γ_q .

Within the finite element:

$$\frac{\partial T}{\partial x_1} = \sum_{i=1}^{LWE} \frac{\partial N_i}{\partial x_1} T_i,$$

$$\frac{\partial T}{\partial x_2} = \sum_{i=1}^{LWE} \frac{\partial T_i}{\partial x_2} u_i.$$

Finally the mimimized functional is replaced by the function of several variables u_i

$$I(u) \approx \frac{1}{2} \lfloor T_1, T_2, T_3, ..., T_{LW} \rfloor \begin{bmatrix} k_{11} & k_{12} & k_{13} & \dots & k_{1LW} \\ k_{21} & k_{22} & k_{23} & & \\ k_{31} & k_{32} & & & \\ \dots & & & & \\ k_{LW1} & & & & k_{LW\,LW} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ \dots \\ T_{LW} \end{Bmatrix} - \lfloor T_1, T_2, T_{3,} \dots, T_{LW} \rfloor \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ \dots \\ b_{LW} \end{Bmatrix}$$

$$I \approx \frac{1}{2} \underset{1\times LW}{\lfloor T \rfloor} \underset{LW\times LW}{[K]} \underset{LW\times 1}{\{T\}} - \underset{1\times LW}{\lfloor T \rfloor} \underset{LW\times 1}{\{b\}}.$$

Minimum- necessary (and sufficient) conditions:

$$\frac{\partial I}{\partial T_i} = 0, \qquad i = 1, \dots, LW. \qquad \Rightarrow \qquad [K]\{T\} = \{b\}$$ **+ Dirichlet b.c.**

B. Thermal stresses – FE equations in the case of mechanical and thermal loads

$$\{\varepsilon\} = \{\varepsilon\}_T + \{\varepsilon\}_s, \qquad \{\varepsilon\}_s = [D]^{-1}\{\sigma\}.$$

[D] – material stiffness matrix [D] $^{-1}$ material flexibility matrix

$$\{\varepsilon\} = [B]\{q\}_e. \quad \text{[B] – element strain matrix}$$

$$\varepsilon_T = \begin{Bmatrix} \alpha_x \Delta T \\ \alpha_y \Delta T \\ \alpha_z \Delta T \\ 0 \\ 0 \\ 0 \end{Bmatrix}_T$$

: $$\{\sigma\} = [D]\{\varepsilon\}_s = [D](\{\varepsilon\} - \{\varepsilon\}_T) = [D]([B]\{q\}_e - \{\varepsilon\}_T). \quad \textbf{(*)}$$

FEM equations derived from the principle of virtual work - taking into account thermal strains

$\{\delta q\}_e$ - vector of nodal virtual displacements of an finite element

$\{\delta\varepsilon\} = [B]\{\delta q\}_e$ - vector of virtual strains within the element

Principle of virtual work for an element

$$\lfloor \delta q \rfloor_e \{F\}_e = \int_{\Omega_e} \lfloor \delta\varepsilon \rfloor \{\sigma\}\, d\Omega_e.$$

$$\lfloor \delta q \rfloor_e \{F\}_e - \lfloor \delta q \rfloor_e \int_{\Omega_e} [B]^T \{\sigma\}\, d\Omega_e = 0,$$

$$\lfloor \delta q \rfloor_e \left(\{F\}_e - \int_{\Omega_e} [B]^T \{\sigma\}\, d\Omega_e \right) = 0.$$

$$\{F\}_e - \int_{\Omega_e} [B]^T \{\sigma\}\, d\Omega_e = 0$$

Using Hooke's law (*) we get:

$$[k]_e \{q\}_e = \{F\}_e + \{F_T\}_e,$$

$$[k]_e = \int_{\Omega_e} [B]^T [D][B]\, d\Omega_e \quad \text{- element stiffness matrix}$$

$$\{F_T\}_e = \int_{\Omega_e} [B]^T [D]\{\varepsilon\}_T\, d\Omega_e$$

- additional vector of nodal forces

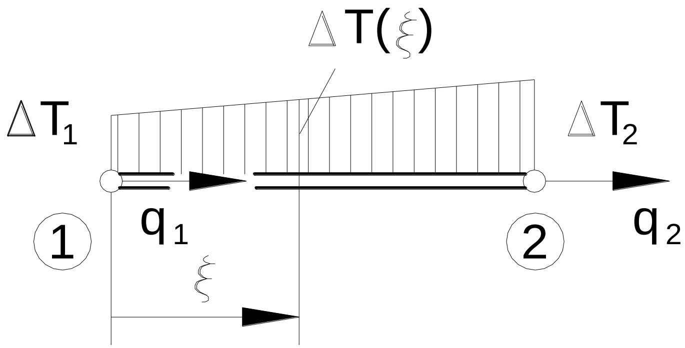
(nodal forces caused by temperature)

FE set of equations for the model

$$[K]\{q\} = \{F\} + \{F_T\}$$

Thermal stresses in rod elements

Basic relations for 2 node rod element



$$u(\xi) = \lfloor N_1, N_2 \rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e, \quad \text{the shape functions :} \quad N_1(\xi) = 1 - \frac{\xi}{l_e}, \quad N_2(\xi) = \frac{\xi}{l_e}.$$

$$\varepsilon(\xi) = \frac{du}{d\xi} = \lfloor N_1', N_2' \rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e, \qquad \sigma(\xi) = E\left(\varepsilon(\xi) - \varepsilon_T\right).$$

$$\Delta T(\xi) = \lfloor N_1(\xi), N_2(\xi) \rfloor \begin{Bmatrix} \Delta T_1 \\ \Delta T_2 \end{Bmatrix}, \qquad \{F_T\}_e = \int_{\Omega_e} [B]^T [D] \{\varepsilon\}_T \, d\Omega_e$$

Vector of thermal nodal forces

$$\{F_T\}_e = \int_{\Omega_e} [B]^T [D] \{\varepsilon\}_T \, d\Omega_e = \int_{\Omega_e} \begin{Bmatrix} N_1' \\ N_2' \end{Bmatrix} \cdot \alpha \cdot E \lfloor N_1, N_2 \rfloor d\Omega_e \begin{Bmatrix} \Delta T_1 \\ \Delta T_2 \end{Bmatrix} =$$

$$= \alpha E A \int_0^{l_e} \begin{bmatrix} N_1' N_1 & N_1' N_2 \\ N_2' N_1 & N_2' N_2 \end{bmatrix} d\xi \begin{Bmatrix} \Delta T_1 \\ \Delta T_2 \end{Bmatrix} = \alpha E A \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} \Delta T_1 \\ \Delta T_2 \end{Bmatrix},$$

$$\{F_T\}_e = \frac{\Delta T_1 + \Delta T_2}{2} \alpha E A \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}.$$

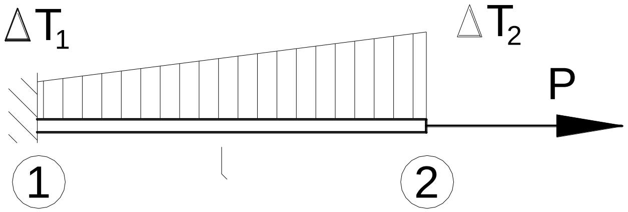
$$[B]^T = \lfloor N_1', N_2' \rfloor^T = \begin{Bmatrix} N' \\ N_2' \end{Bmatrix},$$

$$[D] = E,$$

$$\{\varepsilon\}_T = \varepsilon_T = \alpha \Delta T(\xi) = \alpha \lfloor N_1, N_2 \rfloor \begin{Bmatrix} \Delta T_1 \\ \Delta T_2 \end{Bmatrix}$$

Example

Find the elongation of the rod loaded by the force P and the temperature distribution ΔT



$$[k]_e\{q\}_e=\{F\}_e+\{F_T\}_e,$$

$$\frac{EA}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\end{Bmatrix}=\begin{Bmatrix}F_1\\ P\end{Bmatrix}+\frac{\Delta T_1+\Delta T_2}{2}\begin{Bmatrix}-1\\ 1\end{Bmatrix}\cdot\alpha EA.$$

$$q_1=0$$

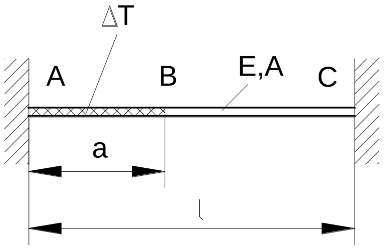
$$\frac{EA}{l}\cdot q_2=P+\frac{\Delta T_1+\Delta T_2}{2}\alpha EA,$$

$$q_2=\frac{Pl}{EA}+\frac{\Delta T_1+\Delta T_2}{2}\alpha l$$

,

Example

Find the stresses in both part of the constrained with heated part AB



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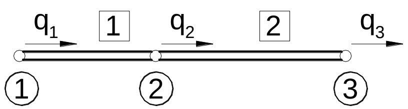


$$EA\begin{bmatrix}\frac{1}{l_1} & -\frac{1}{l_1} & 0\\ -\frac{1}{l_1} & \frac{1}{l_1}+\frac{1}{l_2} & -\frac{1}{l_2}\\ 0 & -\frac{1}{l_2} & \frac{1}{l_2}\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\\ q_3\end{Bmatrix}=\begin{Bmatrix}F_1\\ \alpha EA\Delta T\\ F_3\end{Bmatrix}$$

$q_1=0,\ q_3=0$ => $$EA\left(\frac{l_1+l_2}{l_1l_2}\right)q_2=\alpha EA\Delta T.$$

$$q_2=\alpha\Delta T\frac{l_1l_2}{l_1+l_2}=\alpha\Delta T\frac{a(l-a)}{l},$$

Element 1 : $$\varepsilon_1=\lfloor N_1', N_2'\rfloor\begin{Bmatrix}q_1\\ q_2\end{Bmatrix}=\left\lfloor -\frac{1}{a},\frac{1}{a}\right\rfloor\begin{Bmatrix}0\\ q_2\end{Bmatrix}=\frac{q_2}{a}=\alpha\Delta T\frac{l-a}{l},$$

$$\sigma_1=E(\varepsilon_1-\alpha\Delta T)=-\alpha\Delta TE\frac{a}{l}.$$

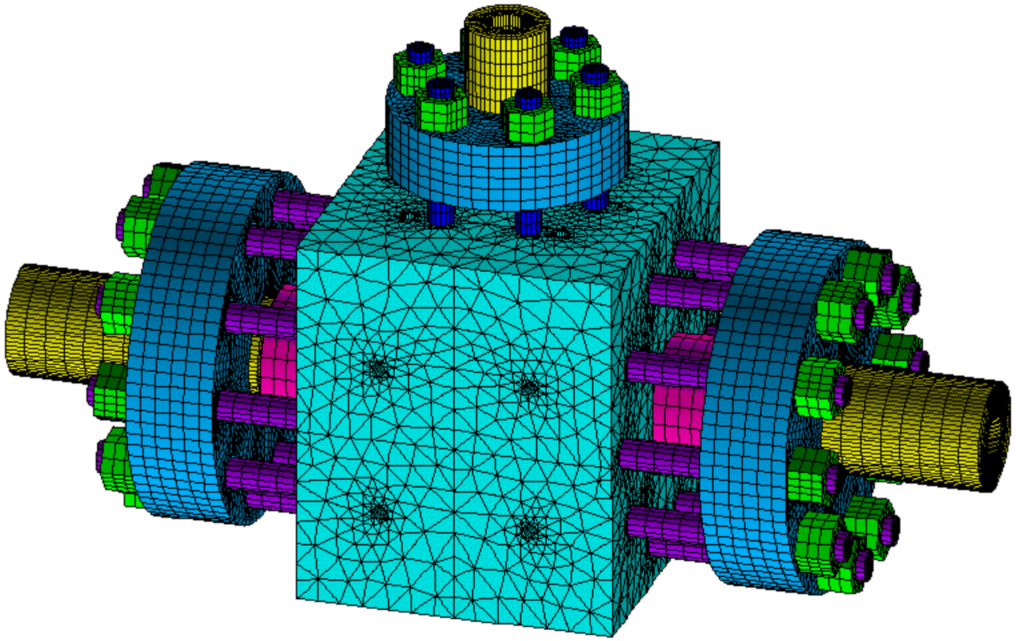
Element 2: $$\varepsilon_2=\lfloor N_1', N_2'\rfloor\begin{Bmatrix}q_2\\ q_3\end{Bmatrix}=\left\lfloor \frac{-1}{l-a},\frac{1}{l-a}\right\rfloor\begin{Bmatrix}q_2\\ 0\end{Bmatrix}=\frac{-q_2}{l-a}=-\alpha\Delta T\frac{a}{l},$$

$$\sigma_2=E(\varepsilon_2-0)=-\alpha\Delta TE\frac{a}{l}.$$

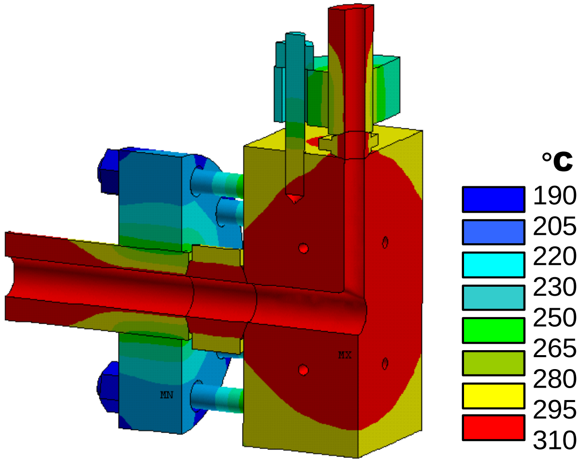
EXAMPLE OF ENGINEERING PROBLEMS OF THERMAL STRESSES

FE analysis of a high pressure T-connection (steady state problem)

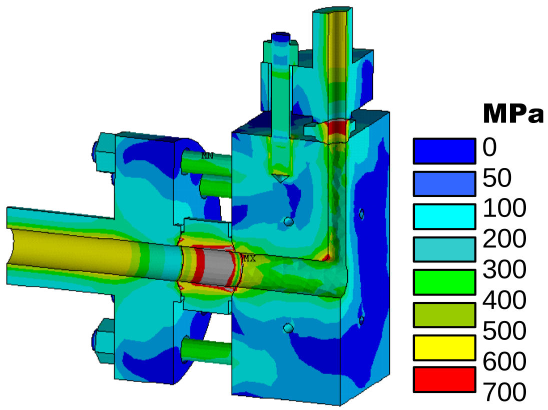
The aim of the analysis was to find stress and strain distribution in a T-connection caused by high internal pressure (2600 at) and the non-uniform temperature. External cooling, assembly procedure (screw pretension), contact and plasticity effects have been included. The project done for ORLEN petrochemical company



FE model



Temperature distribution



Von Mises stress

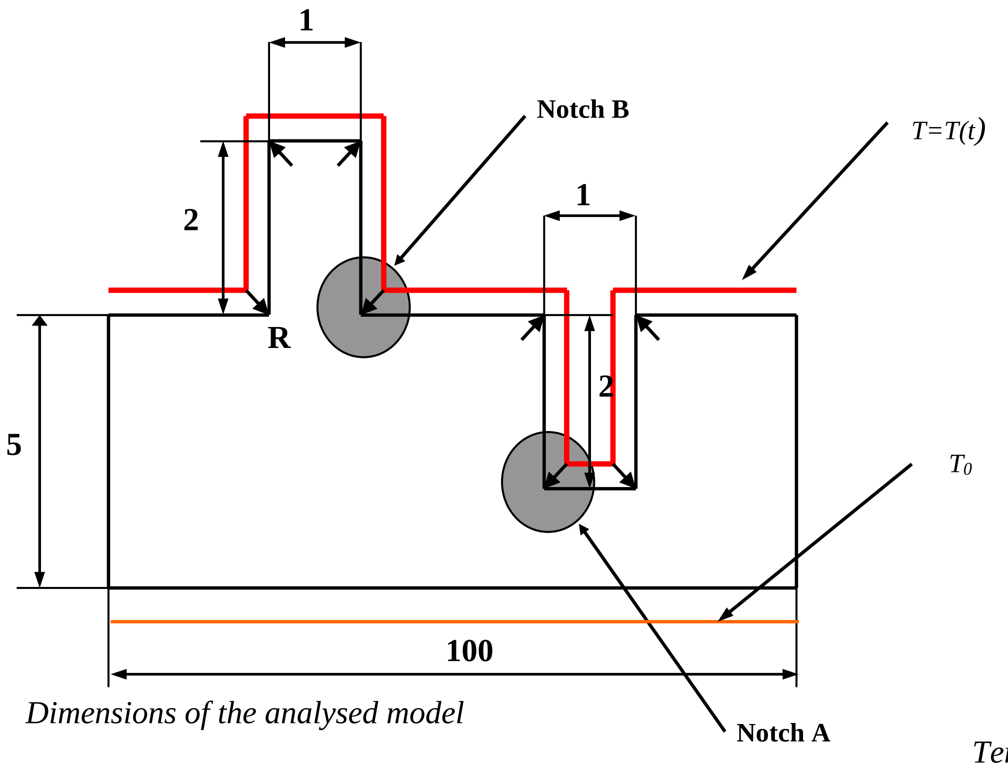
FEM analysis of local stress concentrations caused by impact thermal loads

Analysis description:

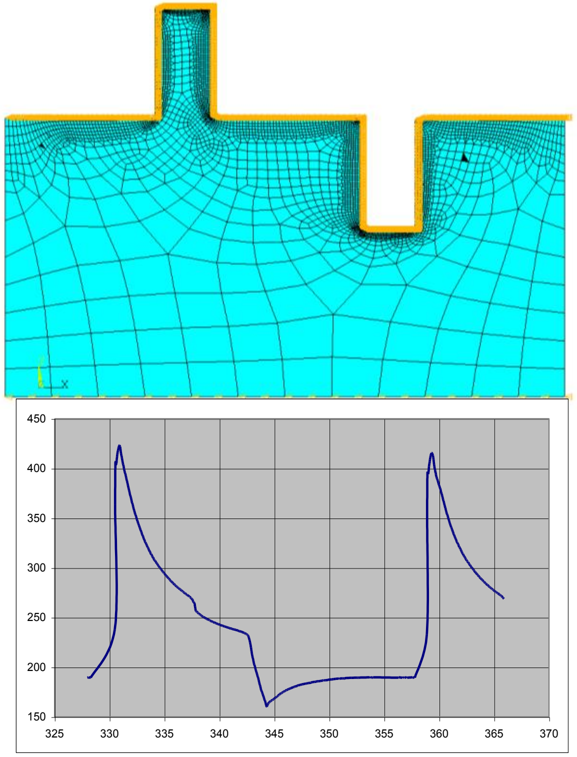
- The main purpose of the analysis was to find stress concentration close to notches during the heating-cooling process.
- Temperature applied at the upper surface, T(t), had been taken from experiments. Analysis were performed for three notch's radius R (R = 0.5, 1.0, 2.0 mm).

Initial and boundary conditions:

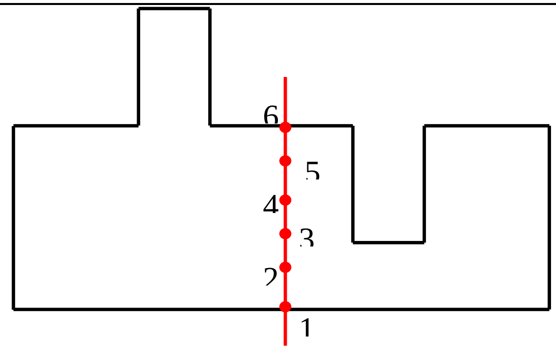
- Temperature of the bottom surface (line in 2D model) $T = T(t_0)$ is constant during the process.
- Reference temperature for structural analysis $T_{REF} = T(t_0)$.Material properties – functions of temperature. No initial stress. Plane strain, Transient analysis, Elasto-plastic material behaviour

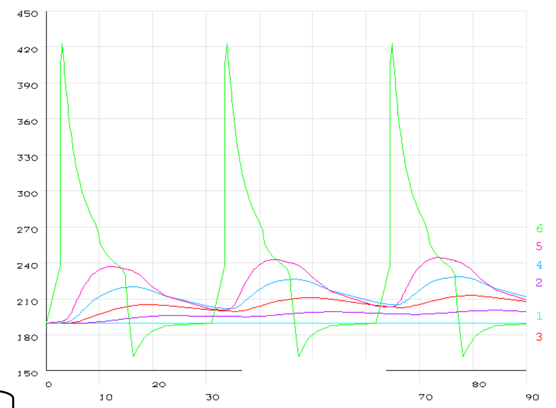


Dimensions of the analysed model

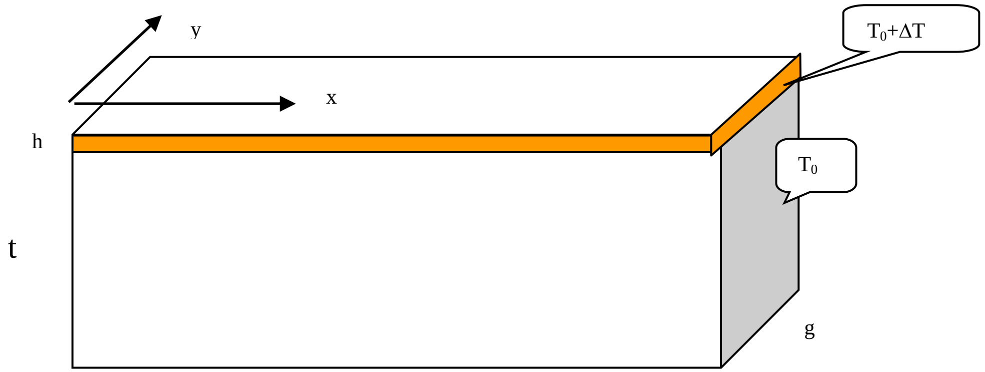


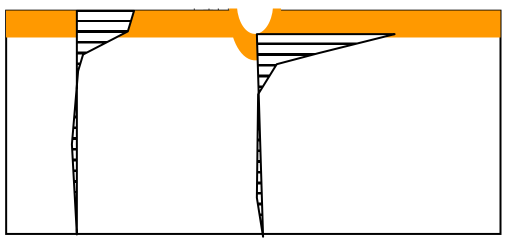
Temperature as the function of time at the surface of the die (from





Temperatures in points 1 to 6 during the process





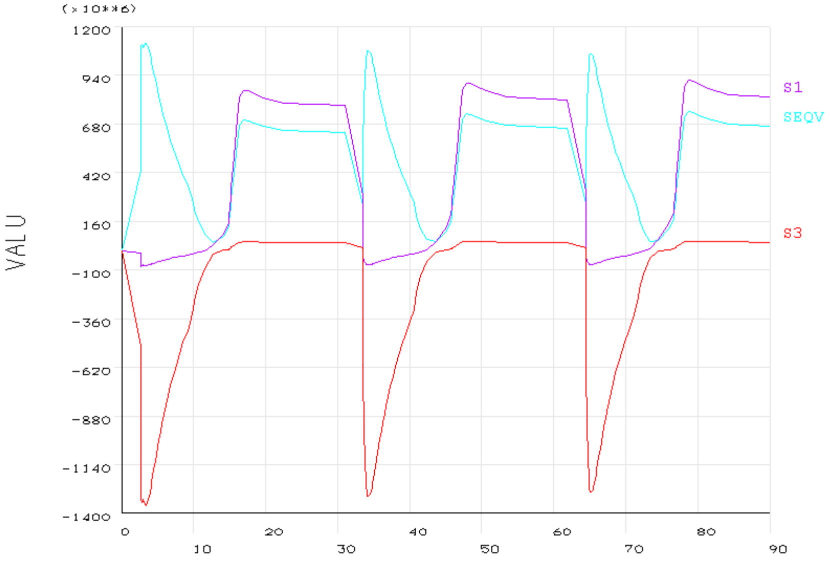
Model of the body with the thin layer subjected to heating- simple analytical considerations

Assuming h<< t within the heated layer is

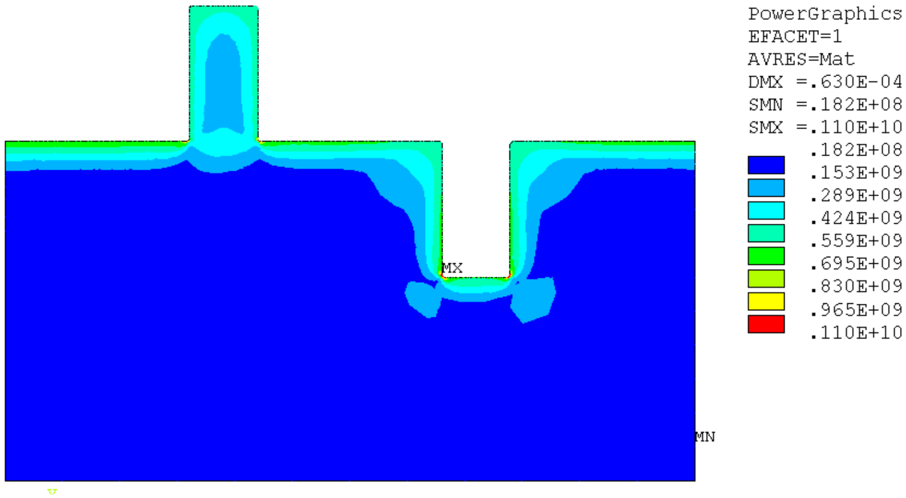
$$\varepsilon_x = 0, \qquad \sigma_x = \sigma_y = \frac{-1}{1-\nu} E\alpha\Delta T,$$
$$\varepsilon_y = 0, \qquad \sigma_z = 0.$$

For $\Delta T \cong 200C$ the result is $\sigma_x = \sigma_y = 685MPa$

$\overline{q} = -\lambda\, \mathrm{grad}(T)$ $q = \alpha_k (T_b - T_s)$ Biot number $Bi = \dfrac{\alpha_k \cdot l}{\lambda}$ gives a simple index of the ratio of the heat transfer resistances *inside of* and *at the surface of* a body. This ratio determines whether or not the temperatures inside a body will vary significantly in space, while the body heats or cools over time, from a thermal gradient applied to its surface.



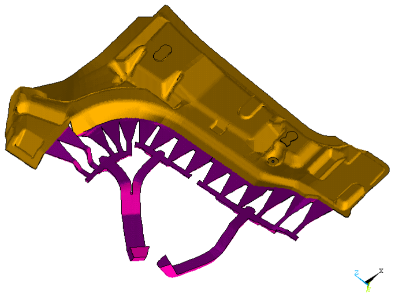
Von Mises (SEQV) and principal stresses at notch A. Notch radius R=0.5 mm



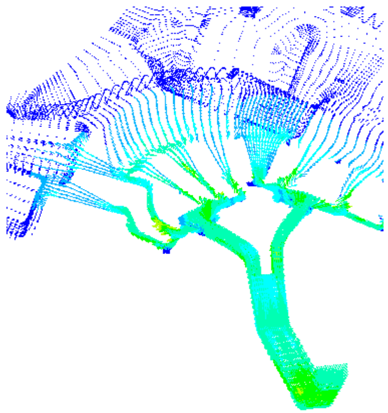
Von Mises stress distribution at time tx. Notch radius R=0.5 mm.

FE analysis of thin-walled elements' deformation during aluminium injection moulding

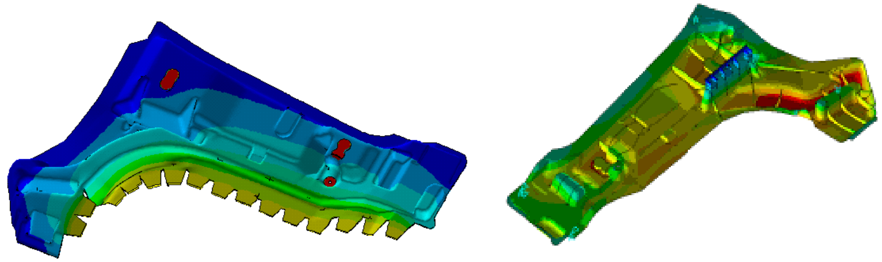
Numerical simulations have been performed to model the process of filling the mould by hot aluminium alloy. The analysis has enabled improvements of the element stiffness diminishing geometrical changes caused by the process. Fluid flow simulation with transient thermal analysis including phase change have been performed, followed by the structural elasto-plastic calculation of residual effects.



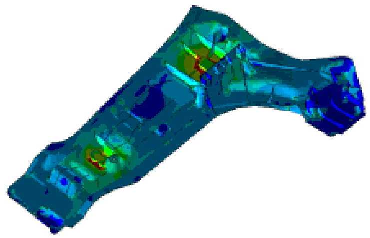
FE model



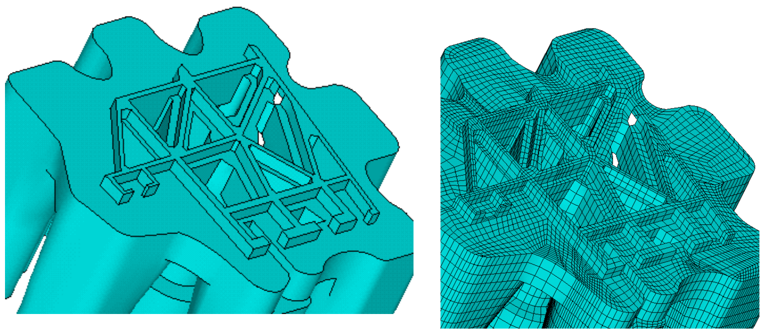
Velocity field during injection process



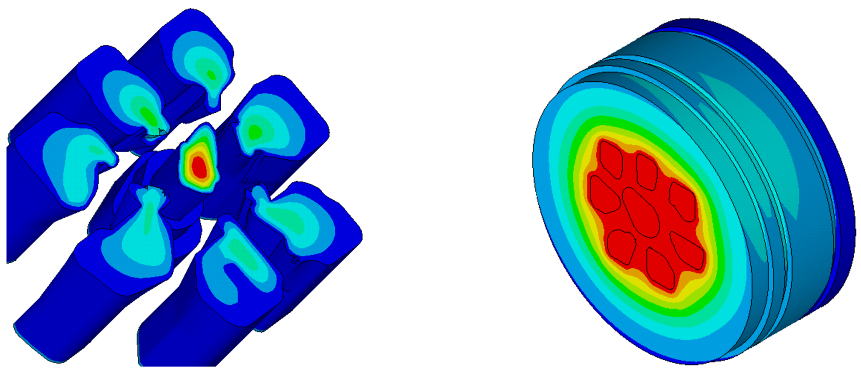
Temperature distribution (cooling effect) and displacements



Residual stress distribution



FE model of the die



Velocity and temperature distribution